PH205: Math Method of Physics

Lecture : Math Method of Physics

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Chapter 1

Prerequisites

1.1 Linear Operators

A linear operator is a function between two vector spaces that preserves the operations of vector addition and scalar multiplication.

$$A(e_i) = \sum_{j=1}^{n} A_{ji} e_j$$

A linear operator maps from one vector space to another. Below the e_i & f_i basis vectors corresponds to vector space V & V'. **Remember:** The difference between linear operator and transformation is that if $V = V'$ the mapping is called operator while if $V \neq V'$ it is linear tranformation in general.

$$V \to V' \implies e_i \to f_i$$

$$i = 1 \ldots n \quad \& \quad j = 1 \ldots n$$

$$\phi(e_i) = \sum_{j=1}^{n} A_{ji} f_j$$

Example: map from n-dimensional vector space to n-dimensional vector space.

$$\phi(e_i) = e_i$$

$$\phi(e_j) = 0, \quad j \neq i$$

This results in a matrix:

$$\begin{pmatrix} 1 & 0 & \ldots \\ 0 & 0 & \ldots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

1.1.1 Projection Operator

A projection operator is a linear operator P from a vector space to itself such that $P^2 = P$.

$$\text{If: } V = \sum_i c_i e_i \quad \& \quad \phi(e_i) = e_i \, \forall i$$

$$\text{Then: } \phi^2(v) = \phi(\phi(v)) = \phi(\phi(c_i e_i)) = \phi(c_i e_i) = c_i e_i = v$$

Example: mapping 2-dimensional to 3-dimensional v-s.

$$\phi(e_1) = f_1 + f_2 + f_3$$

$$\phi(e_2) = f_1 + f_2 - f_3$$

This gives the matrix representation ϕ:

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & -1 \end{pmatrix}$$

The above matrix can be verified if basis are defined as below. You can choose any choice of basis. $\phi(e_i)$ is just matrix multiplication of ϕ with e_i.

$$e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \; e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \; f_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \; f_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \; f_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Matrices occur naturally in representation of linear operator. Matrix algebra are inherited from the rules of linear algebra. Below is the algebra of addition of matrix.

$$A(c_i) = A_{ji}c_j$$

$$B(c_i) = B_{ji}c_j$$

$$(A + B)(e_i) = A(e_i) + B(e_i) = A_{ji}e_j + B_{ji}e_j = (A_{ji} + B_{ji})e_j$$

1.2   Matrix Properties and Operations

1.2.1   Scalar Multiplication and function action on matrix

Scalar multiplication is one of the basic operations defining a vector space.

$$(\lambda A)(e_i) = \lambda A(e_i) = \lambda A_{ji}e_j$$

$$(\lambda A)_{ij} = \lambda A_{ij}$$

The general function which can be expanded in taylor series can act on matrix to give another matrix. If the expansion has infinite terms then it is necessary that each element in matrix addition to converge.

$$e^A = I + A + \frac{A^2}{2!} + \dots$$

Similarly there is $\sin(A)$, $\cos(A)$, $\cosh(A)$, $\sinh(A)$.

$$(1 - A)^{-1} = \frac{1}{1 - A} = I + A + A^2 + \dots$$

Method of computing inverse for matrices close to identity.

1.2.2   Special Matrices and definitions

- $(A^T)_{ij} = A_{ji}$ (Transpose)

- $(A^*)_{ij} = (A_{ij})^*$ (Complex conjugate)

- $(A^\dagger)_{ij} = (A_{ij}^T)^* = (A_{ji})^*$ (Hermitian conjugate)

- $A^T = A$: Symmetric

- $A^T = -A$: Anti-symmetric

- $A^\dagger = A$: Hermitian

- $A^\dagger = -A$: Anti-Hermitian

- $A^* = A$: Real

1.3   Advanced Matrix Concepts

1.3.1   Diagonal and Block Diagonal Matrices

- Diagonal matrix $A_{ij} = 0$ if $i \neq j$.

- Upper triangular matrix $A_{ij} = 0$ if $i > j$.

Matrices can be decomposed into block diagonal.

Block diagonal matrix

A **block diagonal matrix** is a square matrix that is partitioned into smaller square matrices (blocks) along its diagonal, with all off-diagonal blocks being zero matrices. Formally, a block diagonal matrix M can be written as:

$$M = \begin{pmatrix} B_1 & 0 & \cdots & 0 \\ 0 & B_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & B_k \end{pmatrix}$$

where each B_i is a square matrix (block) of size $n_i \times n_i$ and the zeros represent zero matrices of appropriate sizes.

Properties:

- The determinant of a block diagonal matrix is the product of the determinants of its blocks:

$$\det(M) = \prod_{i=1}^{k} \det(B_i)$$

- The trace of a block diagonal matrix is the sum of the traces of its blocks:

$$\mathrm{Tr}(M) = \sum_{i=1}^{k} \mathrm{Tr}(B_i)$$

- The inverse of a block diagonal matrix (if all blocks are invertible) is the block diagonal matrix of the inverses:

$$M^{-1} = \begin{pmatrix} B_1^{-1} & 0 & \cdots & 0 \\ 0 & B_2^{-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & B_k^{-1} \end{pmatrix}$$

Example:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 5 & 6 \end{pmatrix} = \begin{pmatrix} 1 & 0 & & 0 \\ 0 & 2 & & \\ & & 3 & 4 \\ & 0 & 5 & 6 \end{pmatrix}$$

Here, the matrix is block diagonal with two blocks: a 2×2 diagonal block and a 2×2 non-diagonal block.

Block diagonal matrices are useful in linear algebra because they allow us to break down complex problems into smaller, more manageable subproblems, especially when dealing with direct sums of vector spaces or simplifying linear transformations that act independently on subspaces.

1.3.2   Trace and Determinant

Trace of a matrix is sum of diagonal.

$$\mathrm{Tr}(A) = \sum A_{ii}$$

Solving linear equation using matrix representation

$$a_{11}x_1 + a_{12}x_2 = c_1$$

$$a_{21}x_1 + a_{22}x_2 = c_2$$

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$x_1 = \frac{\begin{vmatrix} c_1 & a_{12} \\ c_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}$$

Determinant of a matrix (levi-civita method)

$$\det A = \begin{vmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{vmatrix}$$

$$\epsilon^{123\ldots n} = 1$$

if the order of the index is interchange once like below then the sign changes of each change in order of index

$$\epsilon^{2134\ldots n} = -1$$

if the if any of the index repeat or are equal then

$$\epsilon^{11\ldots i_n} = 0$$

$$\det A = \sum_{i_1,i_2,\ldots,i_n} \epsilon^{i_1 i_2 \ldots i_n} a_{1i_1} a_{2i_2} \ldots a_{ni_n}$$

1.3.3 Applications in Quantum Mechanics

Considering 2 particle fermions

The wave function needs to be anti-symmetric.

$$\psi(x_1)\psi(x_2) - \psi(x_2)\psi(x_1)$$

Slater determinant

$$\begin{vmatrix} \psi_1(x_1) & \psi_2(x_1) & \ldots & \psi_n(x_1) \\ \psi_1(x_2) & \psi_2(x_2) & \ldots & \psi_n(x_2) \\ \vdots & & & \\ \psi_1(x_n) & \psi_2(x_n) & \ldots & \psi_n(x_n) \end{vmatrix} = \begin{vmatrix} \psi_1(x_1) & \psi_2(x_1) \\ \psi_1(x_2) & \psi_2(x_2) \end{vmatrix} \begin{vmatrix} \psi_1(x_1) & \psi_2(x_1) \\ \psi_2(x_1) & \psi_1(x_1) \end{vmatrix}$$

1.3.4 Matrix Inversion

Inverse of a Matrix

$$(A^{-1})_{ij} = \frac{\text{Cofactor } A_{ji}}{|A|}$$

Cofactor $A_{ij} = (-1)^{i+j}$

Gauss-Jordan Matrix Inversion

The **Gauss-Jordan elimination** method can be used to compute the inverse of an invertible matrix A by performing row operations on the augmented matrix $[A \mid I]$ until it becomes $[I \mid A^{-1}]$. The procedure is:

1. Form the augmented matrix $[A \mid I]$, where I is the identity matrix of the same size as A.

2. For each pivot column $j = 1, 2, \ldots, n$:

   (a) Swap rows if necessary to ensure a nonzero pivot entry in row j.

   (b) Scale the pivot row so that the pivot entry becomes 1.

   (c) Use the pivot to eliminate all other entries in column j by adding suitable multiples of the pivot row to the other rows.

3. After processing all n pivot columns, the left half of the augmented matrix will be the identity I, and the right half will be A^{-1}.

Example: Invert a 2×2 matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Form the augmented matrix:

$$[A \mid I] = [\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}].$$

Apply row operations:

$$\begin{aligned}
R_1 &\leftarrow R_1/a, & \text{(make pivot 1)} \\
R_2 &\leftarrow R_2 - c\,R_1, & \text{(zero below pivot)} \\
R_2 &\leftarrow R_2/(d - bc/a), & \text{(make second pivot 1)} \\
R_1 &\leftarrow R_1 - (b/a)\,R_2, & \text{(zero above second pivot)}
\end{aligned}$$

Resulting in $[I \mid A^{-1}]$, where

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

1. Gauss-Jordan Matrix Inversion and Row Operations

To invert a matrix or solve linear systems, the Gauss-Jordan method uses three fundamental row operations:

1. **Multiply a row by a nonzero scalar** (λ):

$$\forall j \quad A_{ij} \to \lambda A_{ij}$$

   This can be represented by multiplying a matrix on the left of A by an elementary matrix R which is the identity matrix except for the ith diagonal entry, replaced with λ:

$$R = \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & \lambda & \\ & & & 1 \end{pmatrix}$$

   Multiplying the above matrix with A on the left is equivalent to Multiplying ith row of A with λ.

2. **Add or Subtract a scalar multiple of one row to another:**

   Subtracting scalar multiple of kth row from ith row

$$A_{ij} \to A_{ij} - \lambda A_{kj} \quad \forall j$$

   The corresponding elementary matrix has $-\lambda$ in the ik position (row i, column k). with all diagonal entries as 1 and other entries as 0.

$$R = \begin{pmatrix} 1 & & & \\ & \ddots & -\lambda & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

3. **Interchange two rows:**
   Swap rows i and k, i.e.,

   $$A_{ij} \leftrightarrow A_{kj}$$

   The corresponding matrix exchanges the identity rows i and k. just making diagonal entries R_{ii} & $R_{kk} = 0$ and R_{ki} & $R_{ik} = 1$ in identity matrix makes it do a row operation of swapping rows when muliplies on left of the matrix

Each of these operations corresponds to left-multiplication by an elementary matrix, and their sequence can systematically convert any invertible matrix to identity, thereby computing its inverse.

2. Matrix Inversion Algorithm

Given an invertible square matrix A, the matrix operation that transforms A into I converts I to A

$$A \to I$$

by applying row operations to both sides:

$$[R_k \dots R_2 R_1]A = I \implies [R_k \dots R_2 R_1] = A^{-1}$$

If A cannot be transformed to I, it is not invertible.

Step-by-Step

1. Choose pivot element (a_{kk}) on the diagonal, make it 1 by division.

2. Eliminate below entries in column k to make zeros using suitable row operations.

3. Repeat for columns $k = 1$ to n to get an upper triangular matrix, then eliminate above diagonal entries to get identity.

Ultimately,

$$R = R_m \dots R_2 R_1 \implies RA = I \implies A^{-1} = R$$

3. Solution of Linear Systems

For $Ax = c$ (A square, c vector):

- **If** $\det A \neq 0$ (i.e., A invertible), **unique solution** exists:

$$x = A^{-1}c$$

- **If** $\det A = 0$:

  - If $Ax = 0$ has a non-trivial solution, then $Ax = c$ can have infinitely many or no solutions (depending on consistency).
  - If $Ax = c$ is consistent (i.e., c in column space of A), infinite solutions exist.
  - If $Ax = c$ is inconsistent, no solutions.

Homogeneous System

If $Ax = 0$:

- If $\det(A) \neq 0 \implies$ only the trivial solution $x = 0$.

- If $\det(A) = 0 \implies$ infinitely many non-trivial solutions possible.

4. Change of Basis and Transformation of Coordinates

Basis Transformation

Suppose e_i is the original basis, and e_i' is the new basis after linear transformation:

$$e_i' = M_{ik}e_k$$

Here, M is invertible (because the bases are linearly independent).
Expressing the inverse transformation:

$$e_k = (M^{-1})_{ki}e_i'$$

Or using index notation:

$$e_i = M_{ij}e_j', \quad e_j' = (M^{-1})_{ji}e_i$$

Vector Components Transformation

A vector X can be represented as $x_j e_j$ or $x_j' e_j'$. When the basis changes:

$$X = x_j e_j = x_i' e_i'$$

Substitute e_i' in terms of e_j:

$$X = x_i'(M_{ik}e_k) = (M^T x')_k e_k$$

Therefore, new components in the original basis are:

$$x_k = M_{ki}^T x_i'$$

Or

$$x' = (M^{-1})^T x$$

This is called *contravariant transformation* for vector components.
The basis vectors themselves transform as:

$$e_i' = M_{ik}e_k$$

which is called *covariant transformation*.

5. Transformation of Operators Under Basis Change

Let A be a matrix representing a linear operator in basis e_j:

$$A(e_j) = A_{ij}e_i$$

Change basis: $e_i' = M_{ik}e_k$.

Matrix Transformation

Let x transform as above:
$$x_i' = (M^{-1})_{ij} x_j$$

Then, in the new basis, the operator A' acting on x':

$$y_i' = A_{ij}' x_j'$$

But $y_i' = M_{ik} y_k$, $x_j' = (M^{-1})_{jl} x_l$.

$$y_i' = M_{ik} y_k = M_{ik} A_{kl} x_l = M_{ik} A_{kl} M_{mj}^{-1} x_j'$$

So the transformation rule for matrices under basis change is:

$$A' = MAM^{-1}$$

This is called a *similarity transformation.*